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ICS 311
Homework 3
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Question for lecture 4

Problem 4-1 on p. 85

Recurrence examples

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T\left(\frac{n}{2}\right) + n^3$.

Answer: We guess that the solution is $T(n) = \Theta(n^3)$. Our method is to prove that $T(n) \le cn^3$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n^3 \\
&= 2c \cdot \left(\frac{n}{2}\right)^3 + n^3 \\
&= \frac{1}{4}cn^3 + n^3 \\
&= cn^3 - \left(\frac{3}{4}c - 1\right) \cdot n^3 \\
&\le cn^3
\end{aligned}
$$

where the last step holds for $c \ge \frac{4}{3}$ and $n > 0$.

b. $T(n) = T\left(\frac{9n}{10}\right) + n$.

Answer: We guess that the solution is $T(n) = \Theta(n)$. Our method is to prove that $T(n) \le cn$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= T\left(\frac{9n}{10}\right) + n \\ &\le \frac{9}{10}cn + n \\ &= cn - \left(\frac{1}{10}c - 1\right) \cdot n \\ &\le cn \end{aligned}$$

where the last step holds for $c \ge 10$ and $n > 0$.

c. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$.

Answer: We guess that the solution is $T(n) = \Theta(n^2 \lg n)$. Our method is to prove that $T(n) \le cn^2 \lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 16T\left(\frac{n}{4}\right) + n^2 \\ &\le \frac{1}{16}c \cdot \left(\frac{n}{4}\right)^2 \cdot \lg\frac{n}{4} + n^2 \\ &= cn^2 \cdot (\lg n - \lg 4) + n^2 \\ &= cn^2 \lg n - (c \lg 4 - 1) \cdot n^2 \\ &\le cn^2 \lg n \end{aligned}$$

where the last step holds for $(c \lg 4 - 1) \cdot n^2 \ge 0$. For example, $c \ge \frac{1}{\lg 4}$ and $n > 0$.

d. $T(n) = 7T\left(\frac{n}{3}\right) + n^2$.

Answer: We guess that the solution is $T(n) = \Theta(n^2)$. Our method is to prove that $T(n) \le cn^2$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 7T\left(\frac{n}{3}\right) + n^2 \\ &\le 7c \cdot \left(\frac{n}{3}\right)^2 + n^2 \\ &= cn^2 - \left(\frac{2}{9}c - 1\right) \cdot n^2 \\ &\le cn^2 \end{aligned}$$

where the last step holds for $\left(\frac{2}{9}c - 1\right) \cdot n^2 \ge 0$. For example, $c \ge \frac{9}{2}$ and $n > 0$.

e. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$.

Answer: We guess that the solution is $T(n) = \Theta(n^{\log_2 7})$. Our method is to prove that $T(n) \le c_1 n^{\log_2 7} - c_2 n^2$ for appropriate choices of the constants $c_1 > 0$ and $c_2 > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 7T\left(\frac{n}{2}\right) + n^2 \\ &\le 7 \cdot \left[c_1 \cdot \left(\frac{n}{2}\right)^{\log_2 7} - c_2 \cdot \left(\frac{n}{2}\right)^2\right] + n^2 \\ &= c_1 n^{\log_2 7} - c_2 n^2 - \left(\frac{3}{4}c_2 - 1\right) \cdot n^2 \\ &\le c_1 n^{\log_2 7} - c_2 n^2 \end{aligned}$$

where the last step holds for $\left(\frac{3}{4}c_2 - 1\right) \cdot n^2 \ge 0$. For example, $c_2 \ge \frac{4}{3}$ and $n > 0$.

f. $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$.

Answer: We guess that the solution is $T(n) = \Theta(n^{1/2} \lg n)$. Our method is to prove that $T(n) \le cn^{1/2} \lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{4}\right) + n^{1/2} \\ &\le 2c \cdot \left(\frac{n}{4}\right)^{1/2} \cdot \lg\frac{n}{4} + n^{1/2} \\ &= cn^{1/2} \lg n - \left(cn^{1/2} \lg 4 - n^{1/2}\right) \\ &\le cn^{1/2} \lg n \end{aligned}$$

where the last step holds for $(c \lg 4 - 1) \cdot n^{1/2} \ge 0$. For example, $c \ge \frac{1}{\lg 4}$ and $n > 0$.

g. $T(n) = T(n-1) + n$.

Answer: In this case, the induction method does not work, so instead we apply the recursion tree method to solve this recurrence.

$$n \left\{ \begin{array}{ccc} T(n) & \Rightarrow & n \\ | & & \\ T(n-1) & \Rightarrow & n-1 \\ | & & \\ T(n-2) & \Rightarrow & n-2 \\ | & & \\ \vdots & & \vdots \\ T(1) & \Rightarrow & 1 \end{array} \right\} = 1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \Theta(n^2)$$

The solution is $T(n) = \Theta(n^2)$.

h. $T(n) = T(\sqrt{n}) + 1$.

Answer: We guess that the solution is $T(n) = \Theta(\lg n)$. Our method is to prove that $T(n) \le c \lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= T(n^{1/2}) + 1 \\ &\le c \lg n^{1/2} + 1 \\ &= \frac{1}{2} c \lg n + 1 \\ &= c \lg n - \left(\frac{1}{2} c \lg n - 1 \right) \\ &\le c \lg n \end{aligned}$$

where the last step holds for $\frac{1}{2} c \lg n - 1 \ge 0$. For example, c and n such that $c \lg n \ge 2$.