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ICS 311
Homework 4
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Question for lecture 5

Problem 4-4 on p. 86

Recurrence examples
I gave solutions to most of the sub problems. But there are three of them to which the master method doesn't apply. Recursion Tree didn't give me a clear enough answer, either. How do I solve sub problems b, d and e?

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 3T\left(\frac{n}{2}\right) + n\lg n$.

Answer: We guess that the solution is $T(n) = \Theta\left(n^{\lg_2 3}\right)$. Our method is to prove that $T(n) \le c_1 n^{\lg_2 3} - c_2 n\lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{2}\right) + n\lg n \\
&\le 3c_1 \cdot \left(\frac{n}{2}\right)^{\lg_2 3} - 3c_2 \cdot \left(\frac{n}{2}\right) \cdot \lg\left(\frac{n}{2}\right) + n\lg n \\
&= c_1 n^{\lg_2 3} - \frac{3}{2}c_2 n \cdot (\lg n - \lg 2) + n\lg n \\
&= c_1 n^{\lg_2 3} - c_2 n\lg n - \left(\frac{1}{2}c_2 n\lg n - \frac{3}{2}c_2 n\lg 2 - n\lg n\right) \\
&= c_1 n^{\lg_2 3} - c_2 n\lg n - n\left[\lg n \cdot \left(\frac{1}{2}c_2 - 1\right) - \frac{3}{2}c_2 \lg 2\right] \\
&\le c_1 n^{\lg_2 3} - c_2 n\lg n
\end{aligned}
$$

where the last step holds for $\lg n \cdot \left(\frac{1}{2}c_2 - 1\right) - \frac{3}{2}c_2 \lg 2 > 0$ e.g, $c_2 = 8$, $n = 2^4 + 1$.

b. $T(n) = 5T\left(\frac{n}{5}\right) + \frac{n}{\lg n}$.

Answer: ?

c. $T(n) = 4T\left(\frac{n}{2}\right) + n^2\sqrt{n}$.

Answer: We guess that the solution is $T(n) = \Theta(n^2\sqrt{n})$. Our method is to prove that $T(n) \le cn^2\sqrt{n}$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n^2\sqrt{n} \\ &\le cn^2\sqrt{\frac{n}{2}} + n^2\sqrt{n} \\ &= cn^2\frac{\sqrt{2n}}{2} + n^2\sqrt{n} \\ &= cn^2\sqrt{n} - \left(\frac{2-\sqrt{2}}{2}cn^2\sqrt{n} - n^2\sqrt{n}\right) \\ &= cn^2\sqrt{n} - n^2\sqrt{n}\cdot\left(\frac{2-\sqrt{2}}{2}c - 1\right) \\ &\le cn^2\sqrt{n} \end{aligned}$$

where the last step holds for $\frac{2-\sqrt{2}}{2}c - 1 > 0$. e.g, $c = 4$ and $n > 0$.

d. $T(n) = 3T\left(\frac{n}{3} + 5\right) + \frac{n}{2}$.

Answer: We guess that the solution is $T(n) = \Theta(n \lg n)$. ?

e. $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$.

Answer: ?

f. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$.

Answer: We guess that the solution is $T(n) = \Theta(n)$. Our method is to prove that $T(n) \le cn$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n \\ &\le \frac{cn}{2} + \frac{cn}{4} + \frac{cn}{8} + n \\ &= cn - n\left(\frac{1}{8}c - 1\right) \\ &\le cn \end{aligned}$$

where the last step holds for $\frac{1}{8}c - 1 > 0$. e.g, $c > 8$ and $n > 0$.

g. $T(n) = T(n-1) + \frac{1}{n}$.

Answer: In this case, the master method does not work, we apply the recursion tree method to solve this recurrence.

$$n \left\updownarrow \begin{array}{ccc} T(n) & \Rightarrow & 1/n \\ | & & \\ T(n-1) & \Rightarrow & 1/(n-1) \\ | & & \\ T(n-2) & \Rightarrow & 1/(n-2) \\ | & & \\ \vdots & & \vdots \\ T(1) & \Rightarrow & 1 \end{array} \right\} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)$$

Based on the formula: $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)$. Therefore the solution is $T(n) = \Theta(\lg n)$.

h. $T(n) = T(n-1) + \lg n$.

Answer: In this case, the master method does not work, we apply the recursion tree method to solve this recurrence.

$$
n \left\{ \begin{array}{ccc} T(n) & \Rightarrow & \lg n \\ | & & \\ T(n-1) & \Rightarrow & \lg(n-1) \\ | & & \\ T(n-2) & \Rightarrow & \lg(n-2) \\ | & & \\ \vdots & & \vdots \\ T(1) & \Rightarrow & \lg 1 \end{array} \right\} = \lg 1 + \lg 2 + \lg 3 + \cdots + \lg n = \sum_{i=1}^{n} \lg i = \Theta(n \lg n)
$$

Based on the formula: $\sum_{i=1}^{n} \lg(i)^c = \Theta\left[n \lg(n)^c\right]$ for nonnegative. Therefore the solution is $T(n) = \Theta(n \lg n)$.

i. $T(n) = T(n-2) + 2\lg n$.

Answer: We guess that the solution is $T(n) = \Theta(n \lg n)$. Our method is to prove that $T(n) < cn \lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &= T(n-2) + 2\lg n \\
&\le c(n-2)\lg(n-2) + 2\lg n \\
&= (cn-2c)\lg n + (cn-2c)\lg(n-2) - (cn-2c)\lg n + 2\lg n \\
&= cn\lg n - \left[cn\lg n + 2\lg n - cn\lg(n-2) + 2c\lg(n-2)\right] \\
&= cn\lg n - \left[cn\lg\frac{n}{n-2} + 2\lg n \cdot (n-2)^c\right] \\
&\le cn\lg n
\end{aligned}
$$

where the last step holds for $cn\lg\frac{n}{n-2} + 2\lg n \cdot (n-2)^c > 0$. e.g, $n > 2$.

j. $T(n) = \sqrt{n}T(\sqrt{n})$.

Answer: We guess that the solution is $T(n) = \Theta(n \lg n)$. Our method is to prove that $T(n) < cn \lg n$ for an appropriate choice of the constant $c > 0$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= \sqrt{n}T(\sqrt{n}) \\ &\le c\sqrt{n} \cdot \sqrt{n} \cdot \lg \sqrt{n} + n \\ &= \frac{1}{2}cn \lg n + n \\ &= cn \lg n - \left(\frac{1}{2}cn \lg n - n\right) \\ &\le cn \lg n \end{aligned}$$

where the last step holds for $\frac{1}{2}c \lg n - 1 > 0$.