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Math 475

Homework #3

March 1, 2010

Section 4.6

Exercise 36-a Let X be a set of n elements. How many different relations on X are there?

Answer: On set X with n elements, we have the following facts.

Number of two different element pairs	$\binom{n}{2}$
Number of relations on two different elements $(a,b) \in R, (b,a) \in R$	$2 \times \binom{n}{2}$
Number of relations including the reflexive ones $(a,a) \in R$	$2 \times \binom{n}{2} + n$
Number of ways to select these relations to form a relation on X	$2^{2\times\binom{n}{2}+n}$

$$2^{2\times\binom{n}{2}+n} = 2^{\frac{2\times n!}{(n-2)!\times 2}+n} = 2^{n(n-1)+n} = 2^{n^2}.$$

b. How many of these relations are reflexive?

Answer: We still have $2 \times \binom{n}{2} + n$ number of relations on element pairs to choose from, but we have to include the reflexive one, $(a,a) \in R$. There are n relations of this kind. Therefore there are

$$2^{\left(2\times\binom{n}{2}+n\right)-n} = 2^{2\times\binom{n}{2}} = 2^{\frac{2\times n!}{(n-2)!\times 2}} = 2^{n(n-1)}.$$

c. How many of these relations are symmetric?

Answer: To select only the symmetric relations on set X with n elements, we have the following facts.

Number of symmetric relation pairs between two elements	$\binom{n}{2}$
Number of relations including the reflexive ones $(a,a) \in R$	$\binom{n}{2} + n$
Number of ways to select these relations to form a relation on X	$2^{\binom{n}{2}+n}$

$$2^{\binom{n}{2}+n} = 2^{\frac{n!}{(n-2)!\times 2}+n} = 2^{\frac{n(n-1)}{2}+n} = 2^{\frac{n(n+1)}{2}}.$$

d. How many of these relations are anti-symmetric?

Answer: To select only the anti-symmetric relations on set X with n elements, we have the following facts.

Number of anti-symmetric relations between two elements	$\binom{n}{2}$
Number of relations including the reflexive ones $(a, a) \in R$	$\binom{n}{2} + n$
Number of ways to select these relations to form a relation on X	$2^{\binom{n}{2}+n}$

$$2^{\binom{n}{2}+n} = 2^{\frac{n!}{(n-2)!\times 2}+n} = 2^{\frac{n(n-1)}{2}+n} = 2^{\frac{n(n+1)}{2}}.$$

e. How many of these relations are reflexive and symmetric?

Answer: To select only the reflexive and symmetric relations on set X with n elements, we have the following facts.

Number of symmetric and reflexive relations between two elements	$\binom{n}{2}$
Number of ways to select these relations to form a relation on X	$2^{\binom{n}{2}}$

$$2^{\binom{n}{2}} = 2^{\frac{n!}{(n-2)!\times 2}} = 2^{\frac{n(n-1)}{2}}.$$

f. How many of these relations are reflexive and anti-symmetric?

Answer: Subtracting the number of relations on X that are reflexive from the number of relations on X that are anti-symmetric would result the number of relations on X that are reflexive and anti-symmetric.

$$2^{\binom{n}{2}} = 2^{\frac{n!}{(n-2)!\times 2}} = 2^{\frac{n(n-1)}{2}}.$$

Exercise 37 Let R' and $R"$ be two partial orders on a set X. Define a new relation R on X by xRy if and only if both $xR'y$ and $xR"y$ hold. Prove that R is also a partial order on X. (R is called the intersection of R' and $R"$.)

Answer: According to the definition of partial order set, we need to prove three properties on the given relation R: reflexive, transitive, and anti-symmetric hold.

<u>*Reflexivity:*</u>

For all $x \in X$, we have $(x, x) \in R'$, since R' is a partial order on X.
For all $x \in X$, we have $(x, x) \in R''$, since R'' is a partial order on X.
$\Rightarrow$ For all $x \in X$, we have $(x, x) \in R$, since both $(x, x) \in R'$ and $(x, x) \in R''$ hold.
Therefore, $(x, x) \in R$ for all $x \in X$. R is a reflexive relation on X.

Transitivity:

Assume $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow (x, y) \in R'$, $(y, z) \in R'$ and $(x, y) \in R''$, $(y, z) \in R''$.

$\Rightarrow (x, z) \in R'$, since R' is a partial order; $(x, z) \in R''$, since R'' is a partial order.

$\Rightarrow (x, z) \in R$, since$(x, z) \in R'$ and $(x, z) \in R''$ both hold.

Therefore, $(x, y) \in R$ $(y, z) \in R$ is followed by $(x, z) \in R$. R is a transitive relation on X.

Anti-symmetry:

Assume $(x, y) \in R$ and $(y, x) \in R$, where $x \neq y$.

$\Rightarrow (x, y) \in R'$, $(y, x) \in R'$ and $(x, y) \in R''$, $(y, x) \in R''$, where $x \neq y$.

$\Rightarrow$ This is a conflict, since R' and R'' are anti-symmetric relations on X.

Therefore, $(x, y) \in R$ is followed by $(y, x) \notin R$. R is an anti-symmetric relation on X.

Exercise 39 Let $(J, \leq)$ be the partially ordered set with $J = \{0,1\}$ and with $0 < 1$. By identifying the subsets of a set X of n elements with the n-tuples of 0's and 1's, prove that the partially ordered set $(X, \subseteq)$ can be identified with the n-fold direct product

$$(J, \leq) \times (J, \leq) \times \cdots \times (J, \leq) \;\; (n\ factors)$$

Answer: By identifying the subset relation of a set with a one element set and an empty set, we have, $(R, \subseteq)$ is a partial order with $R = \{\emptyset, \{x\}\}$, and with $\emptyset \subseteq \{x\}$. There are n elements in set X, so we construct the following relations,

$$\begin{array}{llcl}
R_1: & (\{\emptyset, \{x_1\}\}, \subseteq) & \leftrightarrow & J_1: \;\; (\{0,1\}, \leq) \\
R_2: & (\{\emptyset, \{x_2\}\}, \subseteq) & \leftrightarrow & J_2: \;\; (\{0,1\}, \leq) \\
\vdots & \vdots & \vdots & \vdots \quad \vdots \\
R_n: & (\{\emptyset, \{x_n\}\}, \subseteq) & \leftrightarrow & J_n: \;\; (\{0,1\}, \leq)
\end{array}$$

$$\Rightarrow \quad R_{product} = (R_1, \subseteq) \times (R_2, \subseteq) \times \cdots \times (R_n \subseteq) \quad \leftrightarrow \quad (J_1, \leq) \times (J_2, \leq) \times \cdots \times (J_n \leq)$$

$(J_1, \leq) \times (J_2, \leq) \times \cdots \times (J_n, \leq)$ defines a partial order set on n-tuples of 0's and 1's. So now we need to prove relation, $R_{product} = (R_1, \subseteq) \times (R_2, \subseteq) \times \cdots \times (R_n, \subseteq)$, is the same thing as the subset relation on $X = \{x_1, x_2, \cdots x_n\}$. Therefore, if X_1, and X_2 are subsets of X, the goal is to prove $(X_1, X_2) \in R_{product}$., if and only $X_1 \subseteq X_2$.

If X_1, and X_2 are two subsets of X, and $X_1 \subseteq X_2$, then let $X_1 = \{x_1, x_2, \cdots, x_m\}$, and $X_2 = \{x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_{m+k}\}$. We can then rewrite these two sets in the form of,

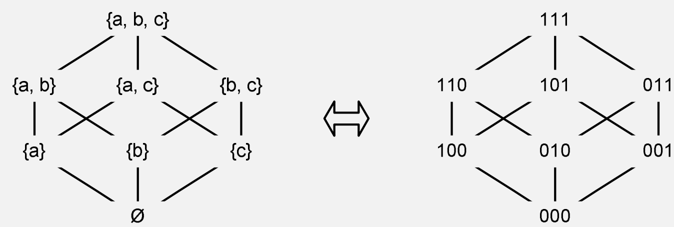
	Column #1	Column #2	Column #3
X_1:	$\{x_1\}, \{x_2\}, \cdots, \{x_m\}$,	$\emptyset_{m+1}, \emptyset_{m+2}, \cdots, \emptyset_{m+k}$,	$\emptyset_{m+k+1}, \cdots, \emptyset_n$
X_2:	$\{x_1\}, \{x_2\}, \cdots, \{x_m\}$,	$\{x_{m+1}\}, \{x_{m+2}\} \;\cdots, \{x_{m+k}\}$,	$\emptyset_{m+k+1}, \cdots, \emptyset_n$

For column #1 $\quad \{x_1\} \subseteq \{x_1\}$; $\{x_2\} \subseteq \{x_2\}$; $\cdots$; $\{x_m\} \subseteq \{x_m\}$,

For column #2	$\emptyset_{m+1} \subseteq \{x_{m+1}\};\ \emptyset_{m+2} \subseteq \{x_2\}; \cdots; \emptyset_{m+k} \subseteq \{x_{m+k}\}$,
For column #3	$\emptyset_{m+k+1} \subseteq \emptyset_{m+k+1};\ \emptyset_{m+k+2} \subseteq \emptyset_{m+k+2}; \cdots; \emptyset_n \subseteq \emptyset_n$

$\Rightarrow$ $(X_1, X_2) \in R_{product}$ according to the definition of relation product.

If $(X_1, X_2) \in R_{product}$ then X_1 and X_2 have to satisfy the form above, therefore $X_1 \subseteq X_2$. Therefore, we've shown that the partially ordered set $(X, \subseteq)$ is identified by the subsets of a set X of n elements with the n-tuples of 0's and 1's. The following example illustrates this corresponding relationship between n-tuples of 0's and 1's with and subset relation. We have $X = \{a, b, c\}$, and 3-tuples of 0's and 1's.



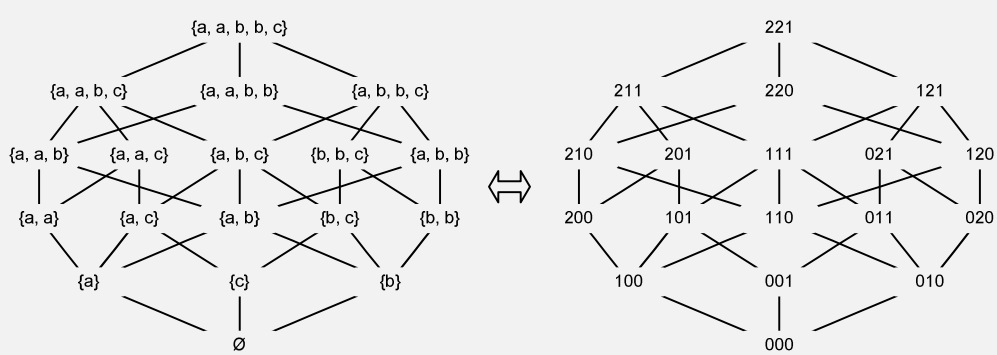
Exercise 40 Generalize Exercise 39 to the multiset of all combinations of the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_m \cdot a_m\}$.

Answer: Let $(J, \leq)$ be the partially ordered set with $J = \{0, 1, \cdots, n\}$ and with $0 < 1 <, \cdots, < n$. The partially ordered set $(X, \subseteq)$ on a multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_m \cdot a_m\}$, can be identified by a m-tuples of 0's, 1's, ..., and n's.

R_1:	$(\{\emptyset, n_1 \cdot \{x_1\}\}, \subseteq)$	$\leftrightarrow$	J_1: $(\{0, 1, \cdots, n_1\}, \leq)$
R_2:	$(\{\emptyset, n_2 \cdot \{x_2\}\}, \subseteq)$	$\leftrightarrow$	J_2: $(\{0, 1, \cdots, n_2\}, \leq)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\vdots$
R_m:	$(\{\emptyset, n_m \cdot \{x_m\}\}, \subseteq)$	$\leftrightarrow$	J_m: $(\{0, 1, \cdots, n_m\}, \leq)$

$\Rightarrow$ $R_{product} = (R_1, \subseteq) \times (R_2, \subseteq) \times \cdots \times (R_m \subseteq)$ $\leftrightarrow$ $(J_1, \leq) \times (J_2, \leq) \times \cdots \times (J_m \leq)$

The proof proceeds the same as the proof in Exercise 39. The following example illustrates this corresponding relationship between n-tuples of 0's, 1's, ..., and n's with and subset relation on a multiset. We have $X = \{a, a, b, b, c\}$, and 3-tuples of 0's, 1's, and 2's.

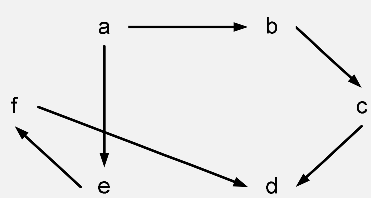


Exercise 42 Describe the cover relation for the partial order $\subseteq$ on the collection $\mathcal{P}(X)$ of an subsets of a set X.

Answer: The cover relation for the partial order $\subseteq$ on set X is the $\subset$ relation. In other words, the transitive closure of $\subset$ together with the reflexive relations on set X makes the relation $\subseteq$.

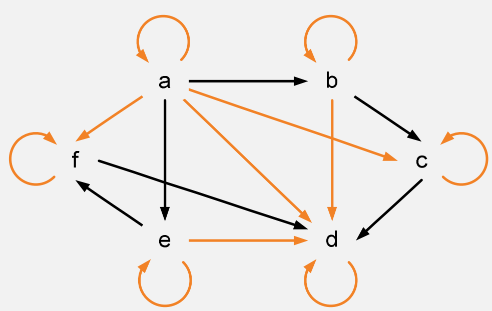
Exercise 43 Let $X = \{a, b, c, d, e, f\}$ and let the relation R on X be defined by aRb, bRc, cRd, aRe, eRf, fRd. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.

Answer: The direct graph representation and the adjacent matrix representation of the given relation on set $X = \{a, b, c, d, e, f\}$ are shown below.



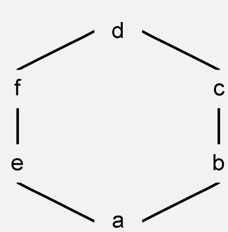
$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The transitive and reflexive closure of the given relation can be represented as the following direct graph.



$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

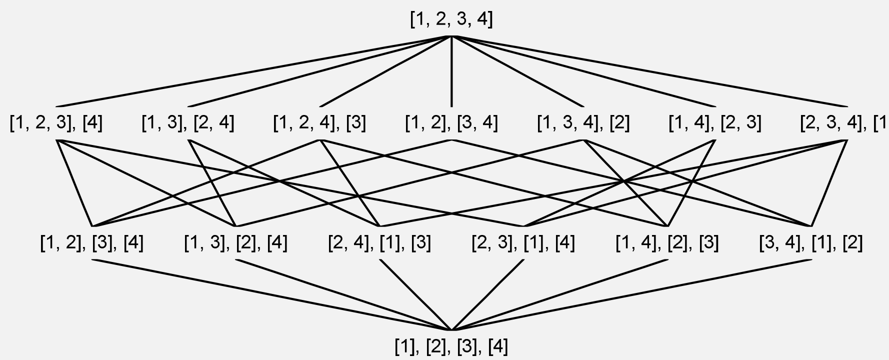
By evaluating the graph, and the adjacency matrix, it's clear that this relation is reflexive, transitive and anti-symmetric. Therefore, the given relation defines a partial order relation on $X = \{a, b, c, d, e, f\}$. The hasse graph of the partial order relation is just the direct graph for the its cover relation. After reordering the vertices and getting rid of the arrows, we have,



The linear extensions are: (a, e, b, f, c, d), (a, e, b, c, f, d), (a, e, f, b, c, d) (a, b, e, f, c, d), (a, b, e, c, f, d), and (a, b, c, e, f, d).

Exercise 47-c Let Π_n denote the set of all partitions of the set $\{1, 2, \cdots, n\}$ into nonempty sets. Given two partitions π and σ in Π_n , define $\pi \leq \sigma$, provided that each part of π is contained in a part of σ. Thus, the partition π can be obtained by partitioning the parts of σ. This relation is usually expressed by saying that π is a refinement of σ. Construct the diagram of $(\Pi_n, \leq)$ for $n = 1, 2, 3,$ and 4.

Answer: The hasse graph representation of this partial order relation on the partitions of set $\{1, 2, 3, 4\}$ is:



Exercise 49-a Prove that the intersection $R \cap S$ of two equivalence relations $R$and S on a set X is also an equivalence relation on X.

Answer: Let's have $R \cap S = T$. To prove that T is a equivalence relation on X, is to show that T satisfies the three properties of the equivalence a relation.

Reflexivity: assume $(x, x) \in T$ is not true for all $x \in X$. Say $(a, a) \notin T = R \cap S$, which is saying either $(a, a) \notin R$ or $(a, a) \notin S$. This is a conflict with the fact that R and S are both reflexive. So the assumption is wrong. Hence, $(x, x) \in T$ is true for all $x \in X$.

Symmetry: assume $(x, y) \in T$, then we have $(x, y) \in R$ and $(x, y) \in S$. $(x, y) \in R \Rightarrow (y, x) \in R$, and $(x, y) \in S \Rightarrow (y, x) \in S$, therefore, $(x, y) \in R \cap S = T$. Hence, $(x, y) \in T$ is followed by $(y, x) \in T$. T is a symmetric relation.

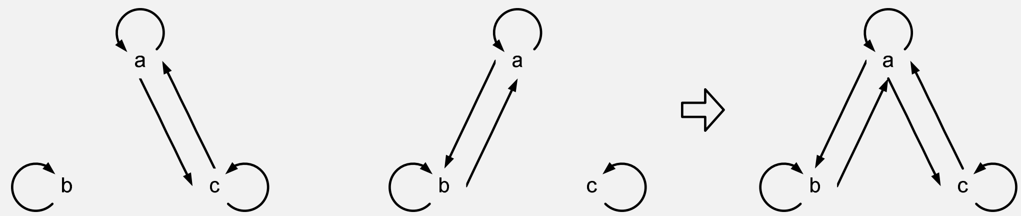
Transitivity: assume $(x, y) \in T$, and $(y, z) \in T$. It follows $(x, y) \in T \Rightarrow (x, y) \in R, (x, y) \in S$. And $(y, z) \in T \Rightarrow (y, z) \in R, (y, z) \in S$. Therefore we have $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$, and $(x, y) \in S, (y, z) \in S \Rightarrow (x, z) \in S$. Since $(y, z) \in R$ and $(y, z) \in S$, $(y, z) \in R \cap S = T$. Hence, $(x, y) \in T$, and $(y, z) \in T$ is followed by $(x, z) \in T$. T is a transitive relation.

So, we've shown that $T = R \cap S$ is a equivalence relation, given both R and S are equivalence relations.

b. Is the union of two equivalence relations on $\boldsymbol{X}$ always an equivalence relation?

Answer: The union of two equivalence relations is not an equivalence relation. Specifically, the transitivity property is not preserved in the union operation. Given below is a counter example:

R: S: $R \cup S$:

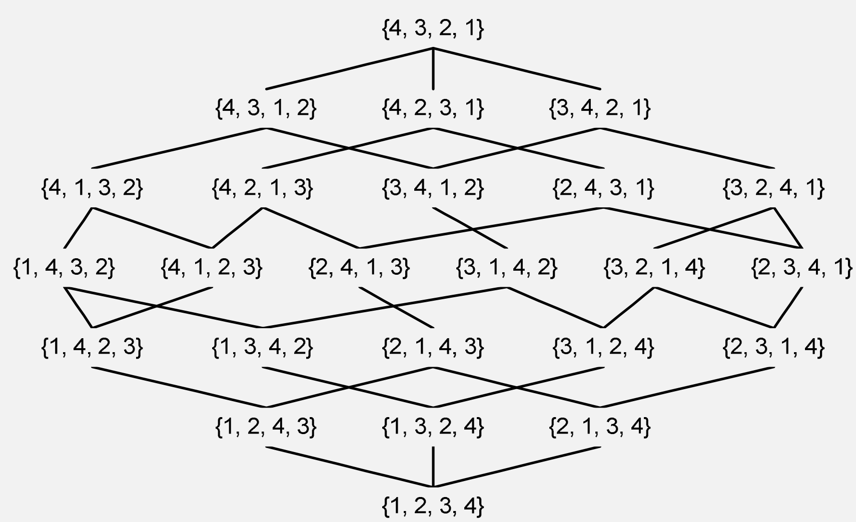


Obviously, $R \cup S$ in the example above is not a transitive relation. $(b, a) \in R \cup S, (a, c) \in R \cup S$, but $(b, c) \notin R \cup S$.

Exercise 51 Let n be a positive integer, and let X_n be the set of $n!$ permutations of $\{1, 2, \cdots, n\}$. Let π and σ be two permutations in X_n, and define $\pi \leq \sigma$ provided that the set of inversions of π is a subset of the set of inversions ofσ. Verify that this defines a partial order on X_n, called the inversion poset. Describe the cover relation for this partial order and then draw the diagram for the inversion poset $(H_4, \leq)$.

Answer: Let π and σ be two permutations in X_n, then in this covering relation, π is covered by σ if the set of inversions of π, $\pi_{inversions}$ is contained by set of inversions ofσ, $\sigma_{inversions}$, $\pi_{inversions} \subset \sigma_{inversions}$. The transitive, reflexive closure of this cover relation gives the inversion poset.

The following hasse graph represent $(H_4, \leq)$.



Exercise 54 Let $(X, \leq)$ be a finite partially ordered set. By Theorem 4.5.2 we know that $(X, \leq)$ has a linear extension. Let a and b be incomparable elements of X. Modify the proof of Theorem 4.5.2 to obtain a linear extension of $(X, \leq)$ such that $a < b$.

Answer: Since a, b are incomparable elements, we can find a partial order $\leq'$ on X that extends from $(X, \leq)$ by adding the relation between element a and b, let $a \leq' b$. According to Theorem 4.5.2, for a finite partial ordered set there is always a linear extension, we can obtain a linear extension on $(X, \leq')$.

Since $(X, \leq')$ is extended from $(X, \leq)$, this linear extension preserves all ordering relations on $(X, \leq)$, therefore, this linear extension is also a linear extension for $(X, \leq)$. At the same time, since $(a \leq' b)$, in this linear extension, we have $a < b$. Hence, we've shown that there exists a linear extension on $(X, \leq)$ that $a < b$, if a and b are incomparable elements of X.

Exercise 55 Use Exercise 54 to prove that a finite partially ordered set is the intersection of all its linear extensions.

Answer: _Way #1:_ According to the previous exercise, for a pair of elements a and b that are incomparable, we can always find a linear extension of $(X, \leq)$ with $a < b$. Let's call this total order relation R_1. Since element a and b are equivalent in this assumption, so we can find a linear extension of $(X, \leq)$ with $a > b$. Let call this total order relation R_2. Since $(a, b) \notin R_2$ and $(b, a) \notin R_1$, then $(a, b) \notin R_1 \cap R_2$ and $(b, a) \notin R_1 \cap R_2$. Both (a, b) and (b, a) are excluded from the intersection of these two linear extensions. Follow the same logic the all relations between incomparable elements will be excluded from the linear extension intersection. The resulting intersection is minimal that contains all of the original relation. In other words, the intersection is $(X, \leq)$ itself.

Way #2: Assume the intersection of all linear extensions of partial order set $(X, \leq)$ contains (a, b), $a < b$ but $(a, b) \notin (X, \leq)$. Since $(a, b) \notin (X, \leq)$, a, b are incomparable, according to the previous exercise, there exist a linear extension of $(X, \leq)$, so that $a > b$. This is a conflict with the assumption that $a < b$ is in the intersection of the linear extensions. Therefore, the intersection of all linear extensions does not contain any extra pair relations that are not in $(X, \leq)$.

Also, the intersection of the linear extensions of a partial order set $(X, \leq)$ can't contain any less that $(X, \leq)$, because every linear extension contains the ordering relations of the comparable elements. These pairwise relations all transact to the intersection. Therefore the intersection of all linear extensions contains at least $(X, \leq)$.

Sum up the conclusions from the two paragraphs above, we conclude that the intersection of all linear extensions on a partial order set $(X, \leq)$ is this partial order set itself.